Vectors

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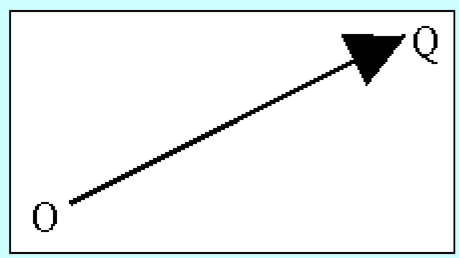
In this tutorial we will examine some of the elementary ideas concerning vectors. The reason for this introduction to vectors is that many concepts in science, for example, displacement, velocity, force, acceleration, have a size or magnitude, but also they have associated with them the idea of a direction. And it is obviously more convenient to represent both quantities by just one symbol. That is the **vector**.

Graphically, a vector is represented by an arrow, defining the direction, and the length of the arrow defines the vector's magnitude. This is shown in Panel 1. . If we denote one end of the arrow by the origin O and the tip of the arrow by Q. Then the vector may be represented algebraically by OQ.



Panel 1

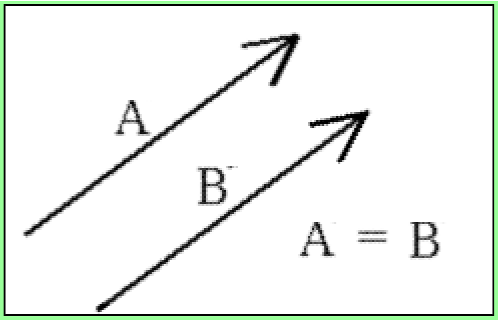
This is often simplified to just $\vec{Q}$ or $\overline{Q}$. The line and arrow above the Q are there to indicate that the symbol represents a vector. Another notation is boldface type as: **Q**.

Note, that since a direction is implied, $OQ \neq QO$. Even though their lengths are identical, their directions are exactly opposite, in fact **OQ = -QO**.

The magnitude of a vector is denoted by absolute value signs around the vector symbol: magnitude of **Q = |Q|**.

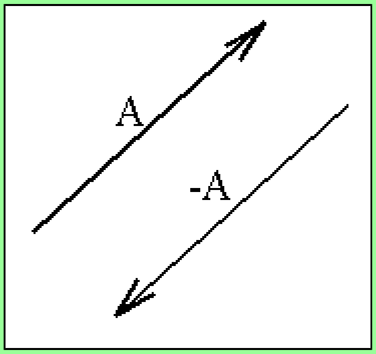
The operation of addition, subtraction and multiplication of ordinary algebra can be extended to vectors with some new definitions and a few new rules. There are two fundamental definitions.

#1 Two vectors, **A** and **B** are equal if they have the same magnitude and direction, regardless of whether they have the same initial points, as shown in Panel 2.



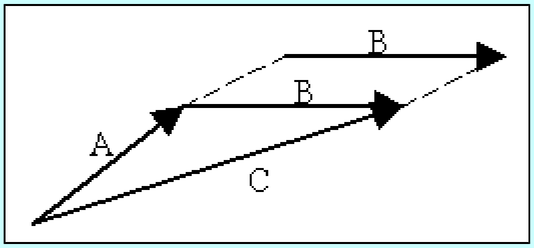
Panel 2

#2 A vector having the same magnitude as **A** but in the opposite direction to **A** is denoted by **-A** , as shown in Panel 3.



Panel 3

We can now define vector addition. The sum of two vectors, **A** and **B**, is a vector **C**, which is obtained by placing the initial point of **B** on the final point of **A**, and then drawing a line from the initial point of **A** to the final point of **B** , as illustrated in Panel 4. This is sometines referred to as the "Tip-to-Tail" method.



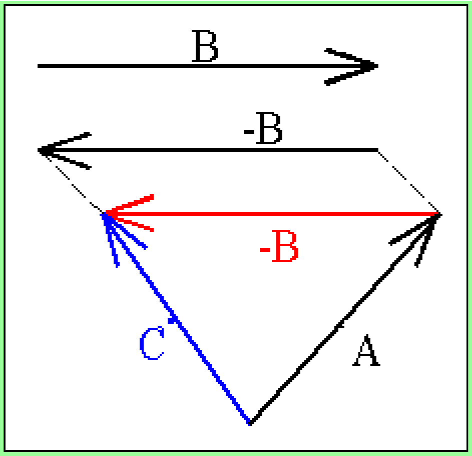
Panel 4

The operation of vector addition as described here can be written as **C = A + B**

This would be a good place to try this simulation on the graphical addition of vectors. *Use the "BACK" buttion to return to this point.*

Vector subtraction is defined in the following way. The difference of two vectors, **A - B** , is a vector **C** that is, **C = A - B**
or **C = A + (-B)**.Thus vector subtraction can be represented as a vector addition.

The graphical representation is shown in Panel 5.
Inspection of the graphical representation shows that we place the initial point of the vector **-B** on the final point the vector **A** , and then draw a line from the initial point of **A** to the final point of **-B** to give the difference **C**.



Panel 5

Any quantity which has a magnitude but no direction associated with it is called a **"scalar"**. For example, speed, mass and temperature.

The product of a scalar, m say, times a vector **A** , is another vector, **B**, where **B** has the same direction as **A** but the magnitude is changed, that is,
|B| = m|**A**|.

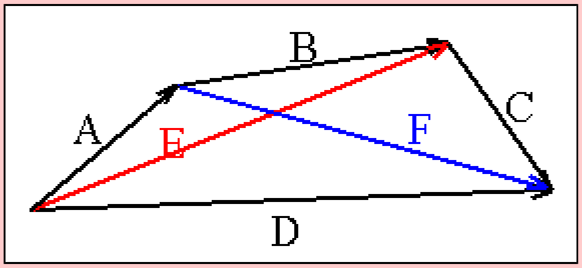
Many of the laws of ordinary algebra hold also for vector algebra. These laws are:

Commutative Law for Addition: A + B = B + A

Associative Law for Addition: A + (B + C) = (A + B) + C

The verification of the Associative law is shown in Panel 6.
If we add **A** and **B** we get a vector **E**. And similarly if **B** is added to **C** , we get **F** .
Now **D** = **E** + **C** = **A** + **F**. Replacing **E** with
(**A** + **B**) and **F** with (**B** + **C**), we get
(A +B) + C = A + (B + C) and we see that the law is verified.

Stop now and make sure that you follow the above proof.



Panel 6

Commutative Law for Multiplication: m**A** = **A**m

Associative Law for Multiplication: (m + n)**A** = m**A** + n**A**, where m and n are two different scalars.

Distributive Law: m**(A + B)** = m**A** + m**B**

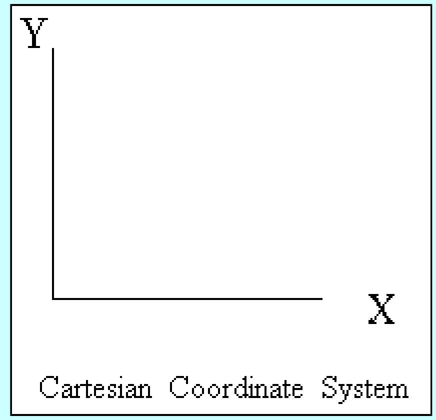
These laws allow the manipulation of vector quantities in much the same way as ordinary algebraic equations.

Vectors can be related to the basic coordinate systems which we use by the introduction of what we call "**unit**

vectors."

A unit vector is one which has a magnitude of 1 and is often indicated by putting a hat (or circumflex) on top of the vector symbol, for example $\text{unit vector} = \hat{a}$, $|\hat{a}| = 1$.The quantity $\hat{a}$ is read as "a hat" or "a unit".

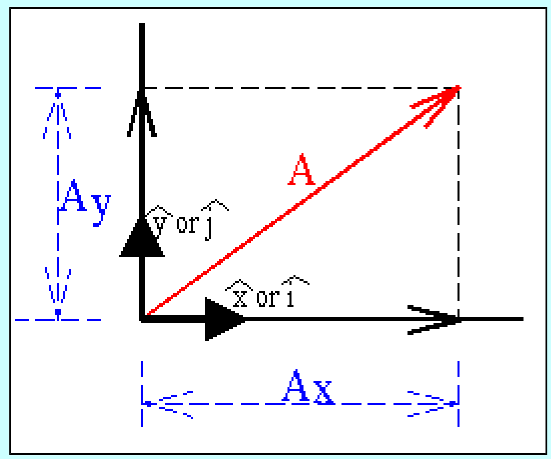
Let us consider the two-dimensional (or x, y)Cartesian Coordinate System, as shown in
Panel 7.



Panel 7

We can define a unit vector in the x-direction by $\hat{x}$ or it is sometimes denoted by $\hat{i}$. Similarly in the y-direction we use $\hat{y}$ or sometimes $\hat{j}$.

Any two-dimensional vector can now be represented by employing multiples of the unit vectors, $\hat{x}$ and $\hat{y}$, as illustrated in Panel 8.

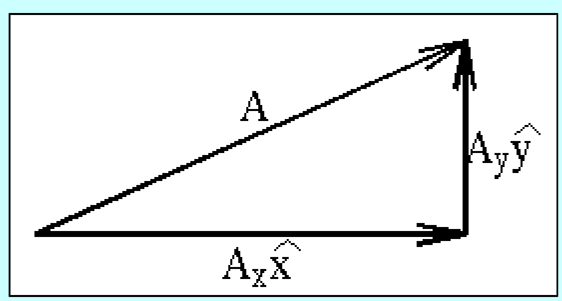


Panel 8

The vector **A** can be represented algebraically by $\mathbf{A} = \mathbf{A_x} + \mathbf{A_y}$. Where $\mathbf{A_x}$ and $\mathbf{A_y}$ are vectors in the x and y directions. If A_x and A_y are the magnitudes of $\mathbf{A_x}$ and $\mathbf{A_y}$, then $A_x\hat{x}$ and $A_y\hat{y}$ are the vector components of **A** in the x and y directions respectively.

The actual operation implied by this is shown in Panel 9.

Remember $\hat{x}$ (or $\hat{i}$) and $\hat{y}$ (or $\hat{j}$) have a magnitude of 1 so they do not alter the length of the vector, they only give it its direction.

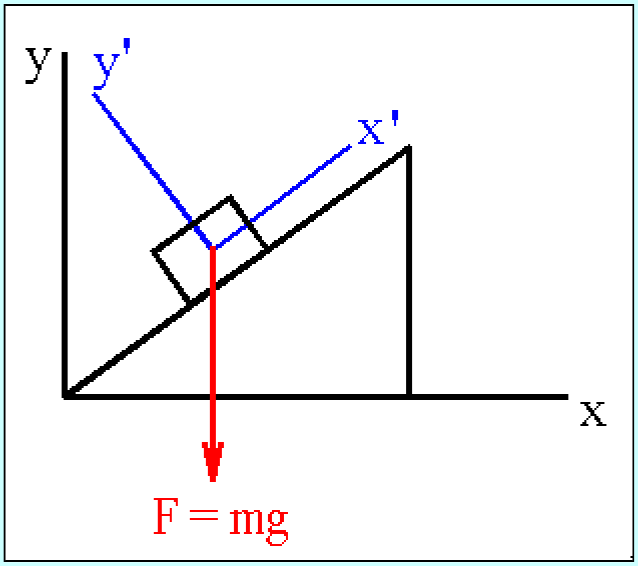


Panel 9

The breaking up of a vector into it's component parts is known as **resolving** a vector. Notice that the representation of **A** by it's components, $A_x\hat{x}$ and $A_y\hat{y}$ is not unique. Depending on the orientation of the coordinate system with respect to the vector in question, it is possible to have more than one set of components.

It is perhaps easier to understand this by having a look at an example.

Consider an object of mass, M, placed on a smooth inclined plane, as shown in Panel 10. The gravitational force acting on the object is **F** = m**g** where **g** is the acceleration due to gravity.



Panel 10

In the unprimed coordinate system, the vector **F** can be written as **F** = $-F_y \hat{y}$, but in the primed coordinate system **F** = $-F_{x'} \hat{x}' + F_{y'} \hat{y}'$. Which representation to use will depend on the particular problem that you are faced with.

For example, if you wish to determine the acceleration of the block down the plane, then you will need the component of the force which acts down the plane. That is, $-F_{x'} \hat{x}'$ which would be equal to the mass times the acceleration.

The breaking up of a vector into it's components, makes the determination of the length of the vector quite simple and straight forward.

Since $\mathbf{A} = A_x \hat{x} + A_y \hat{y}$ then using Pythagorus' Theorem $\mathbf{A} = \sqrt{A_x^2 + A_y^2}$.

For example
If $\mathbf{A} = 3\hat{x} + 4\hat{y}$
then $|\mathbf{A}| = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$.

The resolution of a vector into it's components can be used in the addition and subtraction of vectors.

To illustrate this let us consider an example, what is the sum of the following three vectors?

$\mathbf{A} = A_x\hat{x} + A_y\hat{y}$

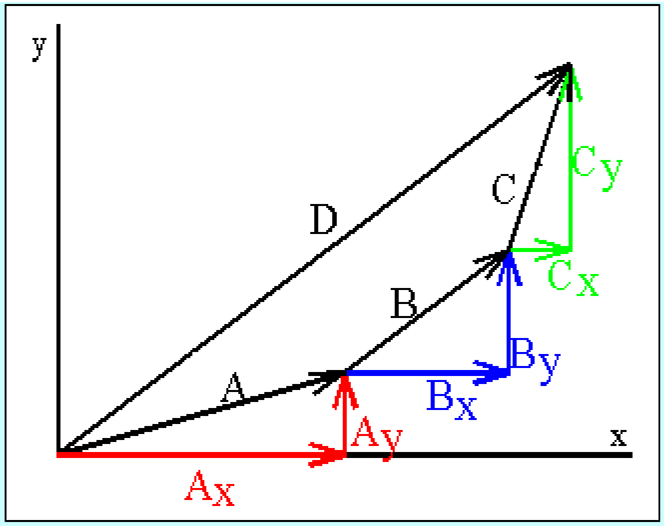
$\mathbf{B} = B_x\hat{x} + B_y\hat{y}$

$\mathbf{C} = C_x\hat{x} + C_y\hat{y}$

By resolving each of these three vectors into their components we see that the result is Panel 11.

$D_x = A_x + B_x + C_x$

$D_y = A_y + B_y + C_y$

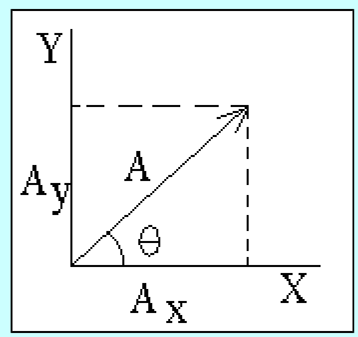


Panel 11

Now you should use this simulation to study the very important topic of the algebraic addition of vectors. *Use the "BACK" buttion to return to this point.*

Very often in vector problems you will know the length, that is, the magnitude of the vector and you will also know the direction of the vector. From these you will need to calculate the Cartesian components, that is, the x and y components.

The situation is illustrated in Panel 12. Let us assume that the magnitude of **A** and the angleq are given; what we wish to know is, what are A_x and A_y?



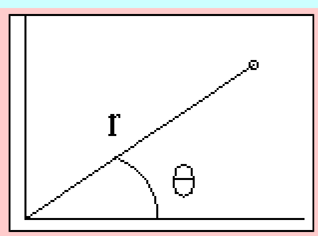
Panel 12

From elementary trigonometry we have, that cosq = A_x/|**A**| therefore A_x = |**A**| cos q, and similarly

A_y = |**A**| cos(90 - q) = |**A**| sinq.

Until now, we have discussed vectors in terms of a Cartesian, that is, an x-y coordinate system. Any of the vectors used in this frame of reference were directed along, or referred to, the coordinate axes. However there is another coordinate system which is very often encountered and that is the **Polar Coordinate System.**

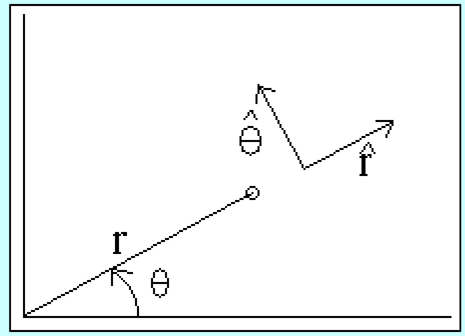
In Polar coordinates one specifies the length of the line and it's orientation with respect to some fixed line. In Panel 13, the position of the dot is specified by it's distance from the origin, that is r, and the position of the line is at some angle q, from a fixed line as indicated. The quantities r and q are known as the **Polar Coordinates** of the point.



Panel 13

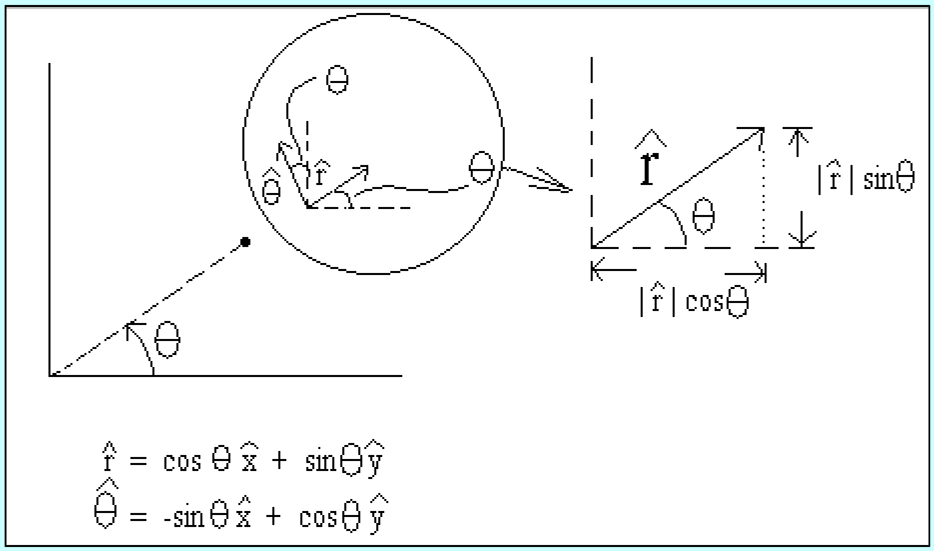
It is possible to define fundamental unit vectors in the Polar Coordinate system in much the same way as for Cartesian coordinates. We require that the unit vectors be perpendicular to one another, and that one unit vector be in the direction of increasing r, and that the other is in the direction of increasing q.

In Panel 14, we have drawn these two unit vectors with the symbols $\hat{r}$ and $\hat{\theta}$. It is clear that there must be a relation between these unit vectors and those of the Cartesian system.



Panel 14

These relationships are given in Panel 15.



Panel 15

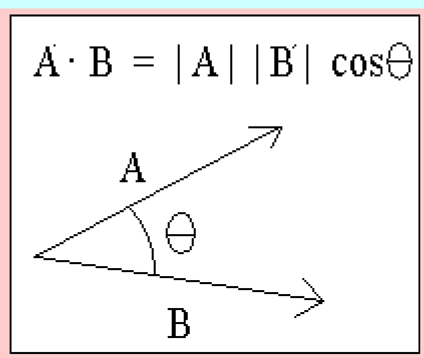
The multiplication of two vectors, is not uniquely defined, in the sense that there is a question as to whether the product will be a vector or not. For this reason there are two types of vector multiplication.

First, the **scalar** or **dot product** of two vectors, which results in a scalar.

And secondly, the **vector** or **cross product** of two vectors, which results in a vector.

In this tutorial we shall discuss only the scalar or dot product.

The scalar product of two vectors, **A** and **B** denoted by **A·B**, is defined as the product of the magnitudes of the vectors times the cosine of the angle between them, as illustrated in Panel 16.



Panel 16

Note that the result of a dot product is a scalar, not a vector.

The rules for scalar products are given in the following list,

$$(1)\ \vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$$
$$(2)\ \vec{A}\cdot(\vec{B}+\vec{C}) = \vec{A}\cdot\vec{B} + \vec{A}\cdot\vec{C}$$
$$(3)\ m(\vec{A}\cdot\vec{B}) = (m\vec{A})\cdot\vec{B} = \vec{A}\cdot(m\vec{B}) = (\vec{A}\cdot\vec{B})\,m$$

.

And in particular we have $\hat{x}\cdot\hat{x} = \hat{y}\cdot\hat{y} = 1$, since the angle between a vector and itself is 0 and the cosine of 0 is 1.

Alternatively, we have $\hat{x}\cdot\hat{y} = 0$, since the angle between $\hat{x}$ and $\hat{y}$ is 90º and the cosine of 90º is 0.

In general then, if **A·B** = 0 and neither the magnitude of **A** nor **B** is 0, then **A** and **B** must be perpendicular.

The definition of the scalar product given earlier, required a knowledge of the magnitude of **A** and **B** , as well as the angle between the two vectors. If we are given the vectors in terms of a Cartesian representation, that is, in terms of $\hat{x}$ and $\hat{y}$, we can use the information to work out the scalar product, without having to determine the angle between the vectors.

If, $\vec{A} = A_x\hat{x} + A_y\hat{y}$,

$\vec{B} = B_x\hat{x} + B_y\hat{y}$

then
$$\vec{A}\cdot\vec{B} = (A_x\hat{x} + A_y\hat{y})\cdot(B_x\hat{x} + B_y\hat{y})$$
$$= A_x\hat{x}\cdot(B_x\hat{x} + B_y\hat{y}) + A_y\hat{y}\cdot(B_x\hat{x} + B_y\hat{y})$$
$$= A_xB_x + A_yB_y$$

.

Because the other terms involved, $\hat{x}\cdot\hat{y} = 0$, as we saw earlier.

Let us do an example. Consider two vectors, $\vec{A} = 2\hat{x} + 2\hat{y}$ and $\vec{B} = 6\hat{x} - 3\hat{y}$. Now what is the angle between these two vectors?

From the definition of scalar products we have $\vec{A}\cdot\vec{B} = |\vec{A}|\,|\vec{B}|\cos\theta$.

$$\therefore\ \cos\theta = \frac{\vec{A}\cdot\vec{B}}{|\vec{A}|\,|\vec{B}|}$$

But .

$$\vec{A}\cdot\vec{B} = 2x6 - 3x2 = 6$$

$$|\vec{A}| = \sqrt{2^2 + 2^2} = \sqrt{8} = 2.83$$

$$|\vec{B}| = \sqrt{6^2 + (-3)^2} = \sqrt{45} = 6.71$$

$$\therefore \cos\theta = \frac{6}{2.83 \times 6.71} = 0.316$$

$$\therefore \theta = 71^{\circ} 36'$$

- This concludes our survey of the elementary properties of vectors, we have concentrated on fundamentals and have restricted ourselves to the discussion of vectors in just two dimensions. Nevertheless, a sound grasp of the ideas presented in this tutorial are absolutely essential for further progress in vector analysis.

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