Answer #338

The answer is (a): the period of the pendulum when swinging at a large angle is *greater* than when it is swung at a small angle, as seen in an mpeg video by clicking your mouse on the photograph below.

By applying Newton's 2nd Law we have:

$$\begin{aligned} F &= m\,a \\ \text{restoring force} &= \text{mass} \cdot \text{linear acceleration} \\ -\cancel{m}\,g\,sin(\theta) &= \cancel{m}\,L\,\ddot{\theta} \end{aligned}$$

where the *restoring force* is the tangential component of gravity. Notice how the restoring force is always towards the center, and is thus negative.

The object's *linear acceleration* on the other hand is obtained by implicit differentiation of the radian measure of a circle twice. Take special note: this assumes the angular displcement is measured in *radians!*

$$\begin{aligned} \text{arclength} &= \text{radius} \cdot \text{angular displacement} \\ \text{linear velocity} &= \text{radius} \cdot \text{angular velocity} \\ \text{linear acceleration} &= \text{radius} \cdot \text{angular acceleration} \end{aligned}$$

Or in other words, more succinctly:

$$
\begin{aligned}
s &= r \cdot \theta \\
v &= r \cdot \dot{\theta} \\
a &= r \cdot \ddot{\theta}
\end{aligned}
$$

which expresses the same as the above but in the generally accepted form. Notice how the radius, *r*, refers to the variable *L* in our situation.

Thus, exact (differential) equation for the *pendulum* is:

$$\ddot{\theta} + \frac{g}{L} sin(\theta) = 0$$

which is a slightly unwieldy equation that requires rather sophisticated mathematics (see below). Since this is *physics*, the above equation is instead usually simplified for the case of small angles:

$$sin(\theta) \approx \theta$$

to become:

$$\ddot{\theta} + \frac{g}{L} \theta = 0$$

which is a rather charming homogeneous, ordinary differential equation of second order.

By inspection, one can see that as the angle becomes larger, the restoring force on the bob becomes relatively less: $sin(\theta) < \theta$. Now, if the restoring force is less, then the pendulum will not be drawn towards the center as strongly, thus leading to a longer period for larger angles of swing. This increase is about 6% for an angle of one radian.

For the brave of heart, the solution of the complete equation for the pendulum, involving large angles, calls for the use of *elliptic integrals*. For the somewhate brave, the integral however can be approximated (yet again) this time through use of infinite series, of which the first three terms are:

$$T = 2\pi\sqrt{\frac{L}{g}}\left(1 + \frac{1}{16}\theta^2 + \frac{11}{3072}\theta^4 + \dots\right)$$

where *T* is the exact period of the pendulum. So in the interests of simplicity the equation is simplified as described above, leading to the *simple pendulum*.

Most intermediate mechanics textbooks discuss the complete solution of the pendulum equation; a number of the references in our demonstration G1-01: EXAMPLES OF SIMPLE HARMONIC MOTION also deal with the solution of the more accurate equation.

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